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ON THICK BOUNDARY LAYERS OVER SLENDER BODIES WITH SOME EFFECTS OF HEAT TRANSFER, MASS TRANSFER AND PRESSURE GRADIENT†

MARTIN H. STEIGER‡ and MARTIN H. BLOOM§

Polytechnic Institute of Brooklyn, Freeport, N. Y.

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Abstract—The integral method is employed to investigate some effects of compressibility, heat transfer, mass transfer and streamwise pressure gradient in boundary layers over slender bodies of revolution, where the boundary layer thickness is not necessarily small compared to the body radius. The results for zero pressure gradient without mass transfer are compared with those of other investigators in the low-speed case and in the adiabatic-surface case. These flows generally produce non-similar profiles. A special case of zero pressure gradient with mass transfer, which yields a similar solution, is solved. For flows with and without pressure gradients and mass transfer, special conditions which produce similar profiles are derived.

NOMENCLATURE

a, body radius, may be function of x;
A, B, C, D, defined by equation (7b);
c_f, skin friction coefficient, $c_f = \dfrac{2\tau_w}{\rho_e u_e^2}$;
G, defined by equation (5);
$\bar{G}$, defined by equation (6c);
J, injection parameter [equation (5)];
m, transformed normal co-ordinate [equation (4a)];
N, transformed normal co-ordinate [equation (4b)];
p pressure;
Re_a, Reynolds number based on body radius;
s, transformed streamwise co-ordinate [equation (5)];
u, v, streamwise and normal velocity component;
x, y, streamwise and normal co-ordinate;
δ, boundary-layer thickness;
δ^*, displacement thickness, $\delta^* = \int_0^\delta (1 - \bar{\rho}\bar{u}) y \, dy$;
δ_m, transformed boundary-layer thickness [equation (4a)];
Δ, defined by equation (7b);
θ, momentum thickness, $\theta = \int_0^\delta \bar{\rho}\bar{u}(1 - \bar{u}) y \, dy$;
Λ, shape factor [equation (6c)];
$\bar{\Lambda}$, defined by equation (8b);
μ, absolute viscosity;
ν, kinematic viscosity, $\nu = \mu/\rho$;
ρ, density;
τ, shear stress.

Subscripts

e, conditions outside boundary layer;
w, refer to conditions at wall;
x, y, N, denotes partial differentiation with respect to indicated variable.

Superscripts

—, denotes dimensionless quantities unless otherwise noted.

$$\bar{u} = \frac{u}{u_e},\ \bar{\rho} = \frac{\rho}{\rho_e},\ \bar{\mu} = \frac{\mu}{\mu_e}.$$

INTRODUCTION

THIS paper treats the viscous, compressible, axially symmetric flow along a body of revolution, paying particular attention to conditions

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‡ Research Associate.

§ Professor, Aeronautical Engineering.

for which the boundary-layer thickness δ is not small compared to the body radius a. An integral method analysis is utilized to investigate both similar and non-similar solutions. Some effects of surface heat transfer, surface mass transfer and streamwise pressure gradients are taken into account.

Work on various aspects of this subject has been done by Seban and Bond [1], Kelly [2], Mark [3], Glauert and Lighthill [4], Probstein and Elliot [5], Pai [6], Yasuhara [7], Bourne and Davies [8], and Bourne *et al.* [9].

Their work has not included effects of mass transfer or pressure gradients. The heat transfer calculations have been confined to incompressible flow [1, 2, 8, 9] or to first-order departures from thin boundary layers [1, 2, 5]. Seban and Bond [1] and Kelly [2] demonstrated that the usual solutions for thin boundary layers begin to incur noticeable errors (underestimates of about 7 per cent in skin-friction coefficient) when $\delta/a \simeq 0{\cdot}16$. They used series expansions in a parameter proportional to $(\delta/a)^2$ to extend the solutions to cases of moderate thickness, with $\delta/a < 1{\cdot}6$, approximately. The solutions in this range are generally non-similar. They obtained temperature and heat-transfer solutions in this range.

Probstein and Elliot [5] used analogous series procedures to treat the compressible case for moderately thick boundary layers ($\delta/a < 1$) over isothermal or insulated cylinders and cones. Yasuhara [7] extended the analysis to deal with bodies of arbitrary shape, again for $\delta/a < 1$.

Mark [3] demonstrated that in incompressible flow a similar solution can be derived for the case of a paraboloid ($a^2 \sim x$) for any magnitude of thickness, provided that streamwise pressure gradients are neglected. Mark's justification for neglecting streamwise pressure gradients was based on a consideration of pressure gradients over paraboloids in inviscid flow, i.e. self-induced pressure gradients. For slender bodies, these are generally small away from the nose region. He did not consider the possible presence of externally caused pressure gradients, such as those due to the impingement of shocks and expansions or due to the immersion of the body in a channel.

Mark's governing differential equation for isobaric flow over a paraboloid was of the form $\bar{f}\bar{f}'' + 2[(1 + \beta\bar{n})\bar{f}'']' = 0$, with boundary conditions $\bar{f}(0) = \bar{f}'(0) = 0$, $\bar{f}'(\infty) = 1$, where $\bar{f}'(\bar{n})$ is the usual velocity ratio, $\bar{n}$ is the similarity variable and β is a constant related to the scale of the body ($a^2/x = 2\nu/u_e\beta^2$). For $\beta \ll 1$, the Blasius equation is obtained. For $\beta \gg 1$, signifying very thick boundary layers, the similarity equation can be manipulated to produce an equation of the form $(f + 1)f'' + nf''' = 0$, with boundary conditions $f(1/2\beta^2) = f'(1/2\beta^2) = 0$, $f'(\infty) = 1$, where dependence upon β appears in the boundary conditions.

Glauert and Lighthill [4] independently obtained the latter differential equation and boundary conditions for a different case, namely, the very thick boundary layer over a cylinder (for which a = constant and $\beta^2 \sim x$, where x is the streamwise co-ordinate). The dependence on x of β, which appears in the boundary conditions, precludes the obtaining of strict, similar solutions in this case. Moreover, Glauert and Lighthill pointed out that the logarithmic variation of f' for small values of n does not permit the transfer of the point of application of the inner boundary condition from $n = 1/2\beta^2$ to $n = 0$. On this basis they did not attempt a solution of the cited equation, but rather started the analysis afresh, with a series expansion in inverse powers of β. They simply noted that the similar solution for paraboloids (β = constant) could be obtained by straightforward numerical integration. Moreover, they pointed out that their expansion procedure for obtaining non-similar solutions for cylinders with $\beta = \beta(x)$ could be extended readily to deal with power-law bodies ($a \sim x^m$, where m is a constant). In fact the solutions up to terms of order $1/\beta^2$ are the same for power-law bodies as for cylinders ($m = 0$).

On the other hand, Mark, following a procedure suggested by Stewartson, carried out a more straightforward type of iterative solution for the second differential equation above, retaining the dependence of the boundary conditions on β, which is constant for the paraboloids he treated. As noted previously, the solutions are strictly similar when β is constant. Mark also obtained a simplified solution of Oseen type by linearizing the convective terms;

this is equivalent to replacing f by n in the bracketed term of the cited differential equation. Since β appears explicitly in the solution, even for $\beta \gg 1$, the solution could have been carried out as readily by the use of the first differential equation above, i.e. the one for $\bar{f}(\bar{n})$.

Both Mark [3] and Glauert and Lighthill [4] carried out integral-method analyses for the condition of zero streamwise pressure gradient to cover the non-similar solutions in the intermediate-thickness range. This range may be defined roughly as $1 < \delta/a < 100$. Glauert and Lighthill apparently did not know of Mark's work, which was not widely available; they dealt only with isobaric incompressible flow. Mark employed the integral method for both incompressible and compressible adiabatic flow, noting the applicability of the well-known Crocco energy-integrals for a Prandtl number of unity. He tested the integral-method solution by a comparison with the exact solutions for the paraboloid in incompressible flow, and by comparisons with the series solutions for the limiting cases of moderate and very large thicknesses in incompressible flow over cylinders. Glauert and Lighthill [4] also employed the latter test. The integral method performed satisfactorily on the basis of these comparisons.

Bourne and Davies [8] carried out the temperature solutions in incompressible flow corresponding to the asymptotic case of extremely large thicknesses analyzed by Glauert and Lighthill [4]. In a further paper, Bourne *et al.* [9] carried out the integral-method temperature analysis in the intermediate-thickness range for incompressible flow, corresponding to the intermediate-range velocity solutions of Glauert and Lighthill [4]. They did not make use of the energy-integral for a Prandtl number of unity or of the Reynolds analogy which is implied by the energy integral.

The present integral-method analysis is similar in approach to those cited above. However, it includes surface heat transfer generally and introduces the elements of surface mass transfer and streamwise pressure gradient. The velocity profile assumed for u/u_e contains logarithmic terms and polynomial terms; both types of function are required for a unified treatment of all cases.

In the pressure-gradient case, further modifications in the profile are required if no restriction is to be placed on the magnitude of the favorable gradients which can be treated. This problem is not peculiar to thick boundary layers but arises also in the analysis of thin boundary layers with strong, favorable pressure gradients. It stems from the fact that polynomial profiles employed in the integral method yield velocity ratios u/u_e greater than unity ("popped" profiles) under strong, favorable pressure gradients even in incompressible, isothermal flow; this result cannot be justified on physical grounds. Therefore, the practice has been to limit the magnitude of the pressure gradients which are treated with polynomial profiles (see, for example, Schlichting [10]) or to employ profiles which preclude "popping" in incompressible, isothermal flow. The latter procedure is discussed more fully by Steiger [11] who utilized a power-law type of profile for this purpose. In this paper the calculations are made with the log-type of velocity profile. In [12], which contains additional analysis of the pressure-gradient case, a few check calculations are made with the power-law type of profile. The pressure gradients for which the log-type profiles are used are limited to those which do not produce popped profiles.

From a technical standpoint, interest in thick boundary layers has increased because of the decreased Reynolds numbers encountered in high-altitude flight and because mass-addition over bodies tends to engender significant boundary-layer thickening. In fact, the coupling of these two conditions can be envisaged readily.

For the constant-wall-temperature and zero-pressure-gradient case, the Prandtl number is assumed equal to unity, which permits the use of the Crocco integral of the energy equation, namely: $H = a + bu$; where a and b are constants.

It is not the intention here to present an exhaustive parametric study of mass-transfer and pressure-gradient effects, but rather to set up the procedure for dealing with these effects and to examine some instructive special cases which do not require very lengthy calculations.

ANALYSIS

The following equations of boundary-layer type are assumed to govern the laminar, viscous, axially symmetric flow along a slender body over which the thickness δ of the viscous layer is not necessarily small compared to the body radius $a(x)$, (see Fig. 1):

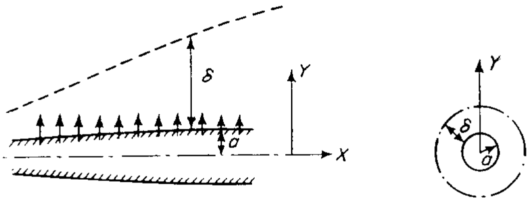

FIG. 1. Schematic of axisymmetric flow along a slender body, with and without injection.

Continuity:

$$(\rho u y)_x + (\rho v y)_y = 0. \tag{1}$$

Momentum:

$$(\rho u^2 y)_x + (\rho u v y)_y = (\mu y u_y)_y - y p_x. \tag{2}$$

The boundary conditions utilized in the integral method are

at $y = a$:

$$u = 0,\ v = v_w,\ y\rho v u_y = (\mu y u_y)_y - y p_x \tag{3a}$$

at $y \to \delta$:

$$u = u_e,\ u_y = u_{yy} = 0. \tag{3b}$$

By operating on equations (1) and (2), and introducing a modified Dorodnitsyn transformation as follows:

$$\bar{\rho} y\, \mathrm{d}y = \mathrm{d}m, \qquad m = \textstyle\int_a^y \bar{\rho} y\, \mathrm{d}y \tag{4a}$$

$$\delta_m N = m, \qquad \bar{\rho} y\, \mathrm{d}y = \delta_m\, \mathrm{d}N,$$

$$(\ )_y = \bar{\rho} \frac{y}{\delta_m} (\ )_N \tag{4b}$$

the following integral differential equation is obtained:

$$\frac{\mathrm{d}\bar{\theta}}{\mathrm{d}s} + \bar{\theta} \frac{\mathrm{d}}{\mathrm{d}s} \ln (\rho_w a^2 u_e^2 + \delta^* \times \frac{\mathrm{d}(\ln u_e)}{\mathrm{d}s} = \frac{2\bar{u}_{N_w}}{G} + J \tag{5}$$

where $\bar{\theta} = \dfrac{G}{2}\displaystyle\int_0^1 \bar{u}(1 - \bar{u})\, \mathrm{d}N = \dfrac{\theta}{\bar{\rho}_w a^2}$,

$$\bar{\delta}^* = \frac{G}{2}\int_0^1 \left(\frac{\rho_e}{\rho} - \bar{u}\right) \mathrm{d}N, \qquad G = \frac{2\delta_m}{\bar{\rho}_w a^2}$$

$$J = \frac{v_w a}{\nu_w}, \qquad \mathrm{d}s = \frac{\nu_w}{u_e a^2}\, \mathrm{d}x.$$

The appropriate boundary conditions in the transformed plane are given by

$$N = 0: \bar{u} = 0,\ \bar{v} = \bar{v}_w,$$
$$\bar{G}\bar{u}_N + \bar{u}_{NN} + \Lambda = 0 \text{ at } y = a \tag{6a}$$

$$N = 1{\cdot}0: \bar{u} = 1,$$
$$\bar{u}_N = \bar{u}_{NN} = 0 \text{ at } y \to \delta \tag{6b}$$

where

$$\bar{G} = G\left[1 - \frac{J}{2}\right];$$

$$\Lambda = \frac{G^2}{4}\frac{\rho_e a^2}{\mu_w}\frac{\mathrm{d}u_e}{\mathrm{d}x} = \frac{G^2}{4\bar{\rho}_w}\frac{\mathrm{d}}{\mathrm{d}s}\ln u_e. \tag{6c}$$

The following velocity profile assumed in terms of N is of the type assumed by Mark [3] and Glauert and Lighthill [4]. It has the advantage of accuracy in the important region near the surface, and is of the type which has yielded satisfactory results over the entire thickness range in the test cases [3, 4] in isobaric, incompressible flow previously discussed in the Introduction.

$$\bar{u} = A \ln (1 + \bar{G}N) + BN^2 + CN^3 + DN^4 \tag{7a}$$

where

$$A = (12 + \Lambda)(1 + \bar{G}^2)/\Delta$$

$$B = -\Lambda/2$$

$$-C\Delta = 4\bar{G}(4\bar{G} + 3) - \Lambda[8(1 + \bar{G})^2 \ln (1 + \bar{G}) - \bar{G}(6\bar{G} + 5)]$$

$$D\Delta = 3\bar{G}(3\bar{G} + 2) - \Lambda[3(1 + \bar{G})^2 \ln (1 + \bar{G}) - \frac{\bar{G}}{2}(5\bar{G} + 4)]$$

$$\Delta = 12(1 + \bar{G})^2 \ln (1 + \bar{G}) - \bar{G}(7\bar{G} + 6). \tag{7b}$$

Fig. 2 illustrates typical profiles for certain values of Λ and $\bar{G}$.

For thin boundary layers $\bar{G} \to 0$, since $G \to 0$, whereas for thick boundary layers $\bar{G} \to 0$ when $J \to 2$. In both cases (7a) is reduced to

$$\bar{u} = 2N - 2N^3 + N^4 + \bar{\Lambda}(N - 3N^2 + 3N^3 - N^4) \quad (8a)$$

where

$$\bar{\Lambda} = \frac{\Lambda}{6} + \frac{\Lambda\bar{G}}{36} + \frac{\bar{G}}{3}. \quad (8b)$$

This reduced profile (8a) is of the same form as that obtained with a fourth-degree-polynomial profile for thin boundary layers with pressure gradients (see [10], p. 208). However, the augmented shape parameter $\bar{\Lambda}$ which appears here is defined differently from the usual shape parameter, when linear terms in $\bar{G}$ are retained. When $\bar{\Lambda} > 12$, values of $u/u_e > 1$ appear in the profile. As discussed in the Introduction, values of $\bar{\Lambda} > 12$ will not be considered here. It is seen that the presence of $\bar{G} > 0$ in (8a) will increase the tendency of the profile to generate values of $u/u_e > 1$ for a given Λ. On the other hand, values of $\bar{G} > 0$ will tend to reduce the tendency toward separation for a given Λ; separation occurs when $\bar{\Lambda} = -12$.

The case of flow with zero pressure gradient, but including surface mass transfer, will now be considered.

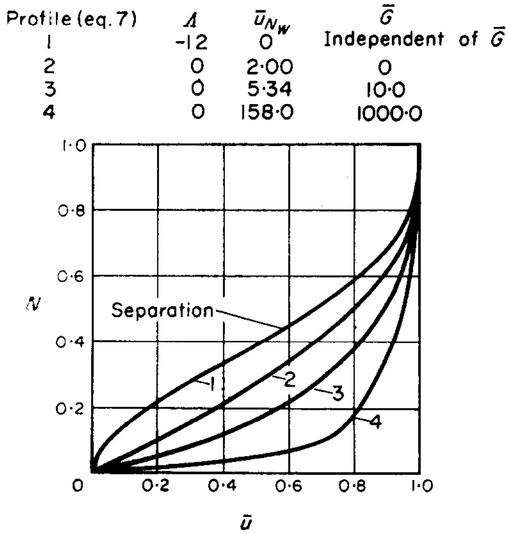

FIG. 2. Velocity distribution in the boundary layer at several longitudinal stations, along a slender body.

Zero pressure gradient with mass transfer

For these conditions ($u_e =$ constant, $\Lambda = 0$) the momentum integral, equation (5), is reduced to

$$\frac{d\bar{\theta}}{ds} + \bar{\theta}\frac{d}{ds}\ln(\rho_w a^2) = \frac{2\bar{u}_{N_w}}{\bar{G}} + J. \quad (9)$$

Similar velocity profiles $\bar{u} = \bar{u}(N)$ are obtained under the condition that $\bar{G} =$ constant. Inspection of (9) shows that this condition is fulfilled if

$$G = 2\delta_m/\bar{\rho}_w a^2 \quad = \text{const.} \quad (10a)$$

$$J = v_w a/\nu_w \quad = \text{const.} \quad (10b)$$

$$b = \frac{d}{ds}\ln(\rho_w a^2) = \text{const.} \quad (10c)$$

It is of interest to derive the mathematically exact similar solutions corresponding to (10) directly from the differential equation (2).†

Integration of (10c) leads to the condition

$$a^2 = \frac{b}{u_e \rho_w}\int_0^x \mu_w\,dx + \left(\frac{\rho_{w0}}{\rho_w}\right)a_0^2 \quad (11)$$

where subscript zero denotes values at an initial station ($x = 0$). For isothermal surfaces, (11) yields *paraboloids.* For $b = 0$, we require $\rho_w a^2 =$ constant. According to (10a) this yields the trivial similar solution $\delta_m =$ constant, or $\delta_m = 0$ if we require the same formal initial condition as for other cases. Therefore, for $b = 0$, we return to (9) to derive solutions. The case of $b = 0$ corresponds to isothermal cylinders or to bodies of the shape $a^2 \sim 1/\rho_w(x)$.

The surface-mass-transfer variation required for similarity follows directly from (10b).

† Briefly, defining

$$\eta = \frac{\sqrt{b}}{\bar{\rho}_w a^2}\int_0^y \bar{\rho} y\,dy$$

and

$$s = \int_0^x \frac{\mu_w}{\rho_w u_e a^2}\,dx,$$

the differential equation (2) becomes

$$ff'' + \left[\frac{\rho\mu}{\rho_w\mu_w}\left(1 + \frac{2}{\sqrt{b}}\int_0^\eta \frac{\rho_w}{\rho}\,d\eta\right)f''\right]' = 0,$$

where primes denote total differentiation with respect to η, $J \equiv -\sqrt{(b)}f(0)$ and $\bar{u} = f'(\eta)$.

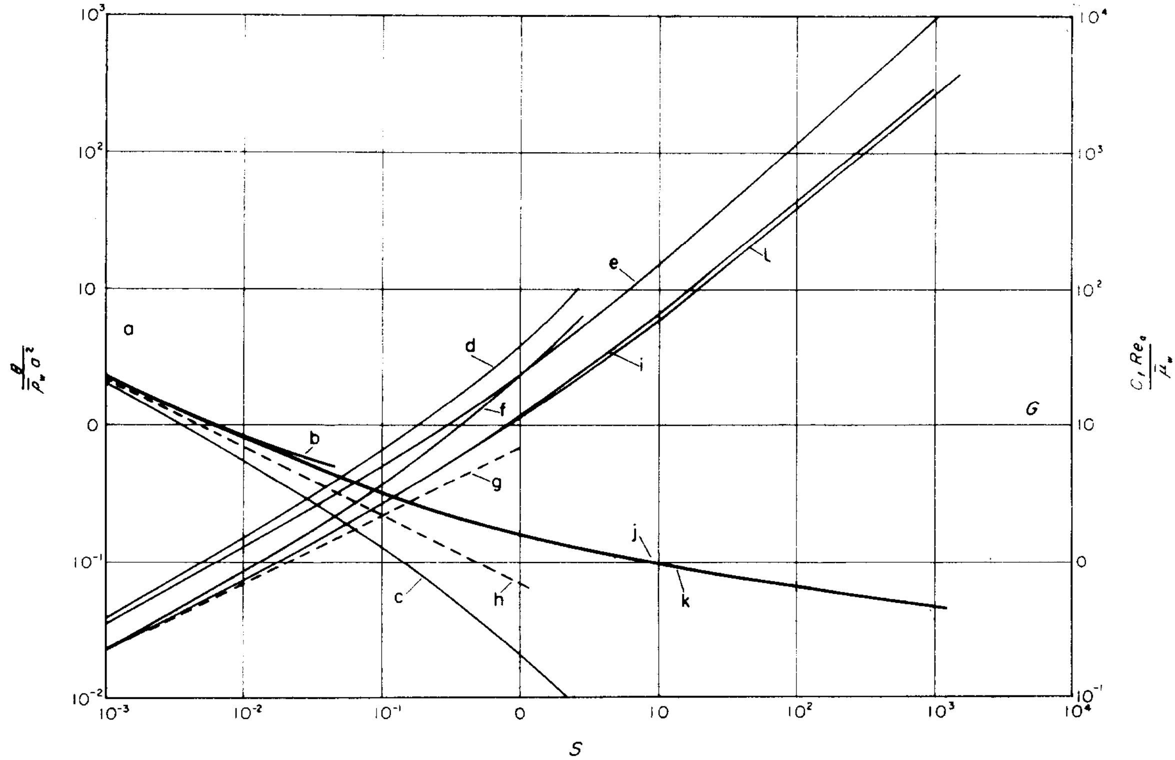

FIG. 3. Variation of parameters for constant pressure and $\rho_w a^2 =$ cont.

a—Present theory $\rho_w a^2 =$ const.;

$$J = \frac{a v_w}{\nu_w} = \text{const.}; \ \Lambda = 0.$$

b—Probstein and Elliot [5] $C_f Re_a$ for $Me \to 0$; $\rho_e = \rho_w$; flow along cylinder.

c—Present theory $\frac{C_f Re_a}{\bar{\mu}_w}$; $J = 2$.

d—Present theory G; $J = 2$.

e—Present theory G; $J = 0$.

f—Present theory $\frac{\theta}{\bar{\rho}_w a^2}$; $J = 2$.

g—Flat plate $\frac{\theta}{\bar{\rho}_w a^2}$; $J = 0$.

h—Flat plate $\frac{C_f Re_a}{\bar{\mu}_w}$; $J = 0$.

i—Glauert–Lighthill [4] θ incompressible flow along cylinder.

j—Glauert–Lighthill [4] $C_f Re_a$ incompressible flow along cylinder.

k—Present theory $\frac{C_f Re_a}{\bar{\mu}_w}$; $J = 0$.

l—Present theory $\frac{\theta}{\bar{\rho}_w a^2}$; $J = 0$.

For the case $b = 0$ and $J =$ constant, the differential equation (9), utilizing the profile (7), can be solved readily for $G(s)$. Results for $J = 0$ and for $J = 2$ (for which $\bar{G} = 0$) are given in Fig. 3 for the complete range of δ/a. In this figure the values obtained here for the thickness parameter $G(s)$, momentum thickness $\theta/\bar{\rho}_w a^2$ and friction coefficient $c_f Re_a/\bar{\mu}_w$ are compared with the incompressible results of Glauert and Lighthill [4] and of Probstein and Elliot [5] for cylinder flow. For isothermal surfaces, the corresponding heat-transfer results can readily be obtained for $Pr = 1$, i.e.

$$q_w/\tau_w = (H_e - h_w)/u_e. \tag{12}$$

The present compressible results are in close agreement with the cited incompressible results. Ref. [12] gives a further comparison of some numerical values which indicate the satisfactory accuracy of the integral method with regard to momentum thickness and skin friction in the limiting cases $\delta/a < 1$ and $\delta/a \gg 1$.

In the special constant-pressure case, for which $\rho_w a^2 =$ constant and for which $J \to 2$ so that $\bar{G} \to 0$, a similar solution with the following relatively simple results is obtained:

$$\bar{\theta} = (36/630)G \tag{13a}$$

$$s = (37/630)\{(G/2) - \ln[1 + (G/2)]\} \tag{13b}$$

$$c_f Re_a/\bar{\mu}_w = 8/G \tag{13c}$$

$$\bar{u} = 2N - 2N^3 + N^4. \tag{13d}$$

The profile (13d) in terms of the transformed variable N is reduced to the usual form for thin isobaric boundary layers. The same result should be obtainable by direct transformation of the differential equations. In this case the explicit effects of viscous thickening are counterbalanced by injection.

The effects of various parameters in the zero-pressure-gradient case with $\rho_w a^2 =$ constant are illustrated in Fig. 3. For zero injection ($J = 0$), the momentum-thickness and skin-friction parameters are larger than their corresponding thin boundary-layer values. The parameter $\theta/\bar{\rho}_w a^2$ varies from $0{\cdot}686\sqrt{s}$, when the viscous layer is much smaller than the body radius ($\delta \ll a$), to $2s$, when the viscous layer is much larger than the body radius ($\delta \gg a$). Analogously, $c_f Re_a/\bar{\mu}_w$ varies from $0{\cdot}686/\sqrt{s}$ to $4/\ln(4s)$. The effect of transverse curvature is the main reason for these trends. An investigation of the velocity distribution in the viscous layer at several stations shows that this curvature acts to augment the effect of a favorable streamwise pressure gradient. This, at least, accounts for the additional increase in the skin-friction parameter. These trends have been more fully discussed in [1–5].

Pressure gradient

The special case for which $J = 2$ warrants attention,† since in this case $\bar{G}$ is zero regardless of the variation of G. Thus the velocity profile assumes the form $\bar{u}(\Lambda, N)$, as indicated by (8), and $\bar{u}_{N_w} = 2 + \Lambda/6$. For $\Lambda =$ constant, the indicated profile is a similar one.

For low-speed, constant-fluid-property flow over a cylinder ($a =$ constant) the solution is especially simple, since (5) and (6c) may be combined to yield an equation in G alone, i.e.

$$G\frac{dG}{ds} = K_1(\Lambda) + GK_2(\Lambda)$$

$$[J = 2;\ \Lambda,\ a,\ \rho \text{ and } \mu \text{ are constant}]. \tag{14a}$$

where

$$K_1(\Lambda) = 2\left(2 + \frac{\Lambda}{6}\right)\left(\frac{G}{\bar{\theta}}\right) - (2 + \delta_i^*/\bar{\theta})4\Lambda \tag{14b}$$

$$K_2(\Lambda) = 2G/\bar{\theta} = 4/\int_0^1 \bar{u}(1 - \bar{u})\,dN. \tag{14c}$$

Equations (14) indicate that $G \ll 1$, $G^2 \simeq 2K_1 s$ (disregarding additive constants), whereas, for $G \gg 1$, $G \simeq K_2 s$. Correspondingly (6c) indicates that, for $G \ll 1$, $u_e \sim s^w \sim x^{w/w+1}$ where $w = 2\Lambda/K_1$; whereas, for $G \gg 1$, $u_e \sim e^{-\Omega/s} \simeq 1 - \Omega/s$ where $\Omega = 4\Lambda/K_2^2$, and $s - \Omega \ln s \simeq s \simeq (\nu/a^2)x$. The skin-friction coefficient is given by the relation

$$c_f Re_a/\bar{\mu}_w = (2\tau_w/\rho_e u_e^2)\left(\frac{\rho_e u_e a}{\mu_e}\right)/\bar{\mu}_w = 4\bar{u}_{N_w}/G \tag{15}$$

which here takes the form

$$c_f Re_a/\bar{\mu}_w = \left(2 + \frac{\Lambda}{6}\right)(4/G). \tag{16}$$

† Additional pressure-gradient effects are discussed in [12].

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Résumé—Les auteurs utilisent la méthode intégrale pour rechercher les effets de la compressibilité, du transfert de chaleur, du transport de masse et du gradient de pression dans les couches limites de corps de révolution, l'épaisseur de la couche limite n'étant pas nécessairement petite devant le rayon du corps. Les résultats pour un gradient de pression nul sans transport de masse sont comparés à ceux d'autres chercheurs, dans le cas où la vitesse est petite et dans le cas d'une surface adiabatique. Ces écoulements n'ont pas, en général, des profils semblables. Les auteurs ont étudié un cas particulier de gradient de pression nul avec transport de masse qui conduit à une solution semblable. Ils ont trouvé les conditions particulières qui mènent à des profils semblables pour des écoulements avec, et sans, gradient de pression et transport de masse.

Zusammenfassung—Mit Hilfe der Integralmethode werden die Einflüsse der Kompressibilität, des Wärme- und Stoffübergangs und des Längsdruckgradienten in der Grenzschicht schlanker Drehkörper untersucht. Dabei muss die Grenzschichtdicke im Vergleich zum Körperradius nicht mehr klein sein. Die Ergebnisse für den Druckgradienten Null ohne Stoffübergang werden mit denen anderer Forscher bei kleinen Geschwindigkeiten und adiabater Oberfläche verglichen. Derartige Strömungen ergeben gewöhnlich nicht-ähnliche Profile. Eine "ähnliche" Lösung liefert ein Sonderfall mit dem Druckgradienten Null und Stoffaustausch; er ist gelöst. Für Strömungen mit und ohne Druckgradient und Stoffübergang ergeben sich unter speziellen Bedingungen ähnliche Profile.

Аннотация—Использован интегральный метод для исследования влияний сжимаемости, переноса тепла, переноса массы и градиента давления на образование пограничного слоя у небольших тел вращения при отсутствии ограничения $\delta \ll a$. Даётся сравнение результатов для нулевого градиента давления при отсутствии переноса массы с результатами, полученными другими исследователями для случаев малых скоростей и адиабатической поверхности. При этом в отличие от обычных случаев с неподобными профилями в рассматриваемом случае получаются подобные решения. Для течений при наличии и отсутствии градиентов давления и переноса массы выведены специальные условия, которые также дают подобные профили.